What Can We Learn about Students' Mathematical Understanding from their Writing?

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Growing from the commognitive framework, this study is concerned with objectification – a special way of talking about mathematical objects, which is key to one's concept formation. The study explores how objectification can be manifested in the discourse on square roots that unfolds in writing. The data comes from a class in a foundation programme where 11 students worked on a specially-designed assignment after intensive engagement with roots. The findings point to the task-dependency of students' written talk about square roots, a struggle to coordinate words and symbols into coherent narratives, and an avoidance of verbal formulations. Theoretical and practical implications are drawn.

Introduction

As mathematics educators, we seem unanimous in our calls for learning mathematics with understanding and for developing deep, strong and well-connected mathematical knowledge among all our students. Yet, more often than not, we produce and absorb stories about students' struggles to reach the desired levels of understanding and knowledge quality. While coming from different countries, classrooms, and even decades, it is rather remarkable how similar such stories can be. Their robustness in space and time evidences the complexity of our educational enterprise; a complexity that requires not only innovative technological and pedagogical approaches for coping with it but also new theoretical lenses for making sense of it. Indeed, what does it mean to "learn mathematics with understanding" and how does "deep, strong and well-connected mathematical knowledge" look like when metaphors are put aside? I propose that answering these questions requires epistemologically solid theories with operational definitions that allow researchers to communicate effectively and be accountable for recommendations that they offer to practitioners.

The discursive framework of Sfard (2008) may be considered as an instance of a theory with the above-mentioned characteristics since it has been acknowledged for providing a comprehensive system of insightful conceptualizations of mathematics and its learning (e.g., Güçler, 2014; Nachlieli & Tabach, 2015; Shinno, 2018). For example, in relation to what is colloquially called "concept understanding", the framework introduces the notion of *objectification* – a special way of talking about mathematical objects as living outside of a human discourse, just as their material congeners. Indeed, mathematically competent discursants might not be aware of how similar their communication about mathematical intangibles and perceptually accessible things-in-the-world sound (as an exercise, notice the structural and syntactical similarities between the sentences "an absolute value of x is the square root of x squared" and "Rexy is the dog of the neighbours"). Such objectified talk, however, is atypical to newcomers who encounter a mathematical object for the first time (Sfard, 2008).

Through multiple examples of small children using numbers as part of their talk, Sfard (2008) illustrates how objectification can be used for analysing the development of their numerical discourse. Yet, it is less clear how this kind of analysis can be conducted in the

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case of mature students who engage with more advanced mathematics and how objectification can manifest itself when one's discourse unfolds in writing.

The study reported in this paper explores this overarching interest in the case of a class of students in a foundation programme who engaged with square roots. Two complementary reasons yielded the decision to focus on this topic. First, in Kontorovich (2018a), I position roots as a cross-curricular concept that students encounter several times in their mathematics education journey, but the definitions (and consequent properties) of roots can change radically from one encounter to another (e.g., real roots in arithmetic and algebra, root functions in calculus, complex roots). Hence, different cohorts of school teachers and university lecturers might benefit from an evidence-based picture of students' engagement with this rarely explored concept (see Shinno, 2018 for an exception). Second, as it probably is in most cases where an algebraic entity is under consideration, symbolism is ingrained in the discourse on square roots. By untangling the relation between symbolic and verbal counterparts of students' discourse, this study offers a theoretical refinement of the notion of objectification as it emerges from a written communicational medium.

Commognitive Framework in a Nutshell

In her framework, Sfard (2008) positions mathematics as a collectively maintained discourse, which is in a constant flux because its participants differ in their aims, thinking, and commitment. A participation in the discourse requires an individual to communicate with others and with oneself, with the latter being defined as *thinking*. Hence, the neologism *commognition* is often used in regard to the framework as a combination of "communication" and "cognition".

Unlike colloquial discourses that often revolve around material entities, the commognitive framework posits that mathematical objects are discursive – i.e., they come into being through humans' words, symbols, narratives, and routines. As it has been mentioned in Introduction, the discursive nature of mathematics is easy to miss due to the objectified ways competent discursants communicate about its objects. Specifically, the ubiquitous mechanisms of *reification* and *alienation* make the objects sound as capable of a mind-independent existence: "Reification is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44). Operatively speaking, reification is manifested in one's usage of mathematical words as nouns rather than as adjectives and verbs (e.g., "the square root of 9 gives us 3" is a reification of "after extracting the square root from 9 we got 3[1]"). Alienation erases the human agency from a narrative, which results in impersonal sentences (e.g., "the square root of 9 is 3"). To clarify, Sfard (2008) argues that objectification is an unavoidable feature of human discourses, which allows them to grow in communicative and practical effectiveness. Indeed, notice how reification and alienation compress the sentences in the examples into the concise $\sqrt{9}=3$, which can be handled now as an object. In this way, the discursive changes that someone goes through when communicating about mathematical objects are interpreted by commognitive analysts as tangible evidence of this someone's *learning*.

[1] As surprising as it may sound, literate discourses on square roots vary in different countries (Kontorovich, 2018b). According to the Israeli school curriculum, in the field of reals, a number b is a square root of a if $b^2=a$. The expression "*the* square root" and '√'-symbol are used to refer to non-negative roots only. In this way, the symbolic statement $\sqrt{9}=3$ is correct when it represents the function $f(x)=\sqrt{x}$ at $x=9$ and when it is considered as an operation between numbers.

The notion of mathematical object has been used extensively until now but it has not been operationalized yet. The commognitive operationalization uses the notion of *signifier* – a perceptually accessible entity that can be realized into another signifier, which in turn, can be realized into another signifier, and so on, forming a *realization tree*. Thus, a *discursive mathematical object* signified by S is a personal construct consisting of the realization tree of S within a particular discourse. Let us consider an example: a student was assigned with a question "Give the square root of x^2" and she wrote "$\sqrt{x^2}$". From the commognitive standpoint, the production of the answer involved at least three signifiers that were successively realized by the student: the written "square root", the phonetical "skweə ruːt" and the symbolic "$\sqrt{x^2}$". Our dissatisfaction with the student's answer may be captured in terms of the "length" and "richness" of her realization tree – two criteria that attest to the quality of her discourse. Another quality criterion might pertain to the situatedness of her realization trees – i.e., how stable they are when the same signifiers are mentioned in situations involving different interlocutors and interactions.

Research Aim

The previous section shows that the notion of objectification can be analytically powerful in cases where a teacher or researcher has access to students' oral talk. Yet, in many classrooms, written communication is the accepted medium for students to demonstrate their mathematical proficiency and for a teacher to provide a constructive feedback. Then, the aim of this study is to characterize how reification, alienation, and objectification can manifest themselves in students' written narratives; specifically, through the use of symbols and words.

Method

The data for this study comes from a class of 11 students, who at the time of data collection, were enrolled in a foundation programme affiliated with a large technological university in Israel. The participants were eighteen- and nineteen-year-olds who finished school with the minimal mathematical requirements of the national educational system. The participants can be considered as typical students of a foundation programme since it is intended for those, whose school achievements are not sufficient for getting accepted to the universities and faculties of their choice. The programme provides its students with an opportunity to apply for academic studies based on their achievements in intense foundation courses in mathematics, physics, and English. As part of the mathematics course, ten teaching hours are dedicated to roots, where students engage in solving hundreds of questions on the topic. The data was collected nearly three weeks after the topic was covered.

I collected the data with an assignment consisting of 17 tasks. In the first 15, the students were asked to extract square roots from numbers and parameters. Due to a special choice of numerical values and algebraic structures (see Figures 1a and 2a for examples) the computational complexity of these tasks was substantially lower than the ones that the students encountered as part of their programme studies beforehand. The last two tasks were concerned with the radical symbol and the definition of a square root. These two tasks, as well as the request to explain their symbolic manipulations in writing, might have been less familiar to the students. I distributed the assignment in one of the mathematics lessons and asked the students to work on it individually. While their work was not time-limited, all students submitted their assignments in 25 minutes.

The principles of commognitive research were employed for analysing students' responses (Sfard, 2008). Specifically, I systematically contrasted students' use of symbols and words and traced the changes that their narratives underwent throughout the assignment. This analysis resulted in categories that were interpreted with the theoretical framework and led to Findings. The excerpts that I use to illustrate the findings in the next section are translations of the responses that the students provided. In translation, I aimed at preserving the idiosyncratic structure of students' narratives even when it came at a cost of violating the rules of English grammar.

Findings

Due to space limitation, I structure this section around two findings that emerged from the analysis of students' assignments: the struggles with objectification and words, and the task-dependency of objectification. Excerpts from the assignments of Anna and Betty (pseudonyms) are used to illustrate the findings.

Struggles with Objectification and Words

Figure 1a illustrates a square-rooting routine that was identified in Anna's responses to the tasks in which square roots were extracted from squared numbers and parameters (i.e. $\sqrt{\blacksquare^2}$): she started with converting the radical symbol to the power of half, followed with reducing the powers to 1, and concluded with the number or parameter that has been squared initially. In this way, the length of Anna's symbolic strings was more or less the same in all the tasks. Accordingly, while she showed robustness when simplifying '$\sqrt{x^2}$' into 'x', Anna's systematic adherence to compound symbolic chains indicates that there is no immediate link between the two signifiers in her realization tree and she needs a multi-step procedure for converting one symbol into another. In contrast, when roots were extracted from square numbers (e.g., $\sqrt{169}$), Anna provided immediate answers without capturing any procedure in a form of a written text.

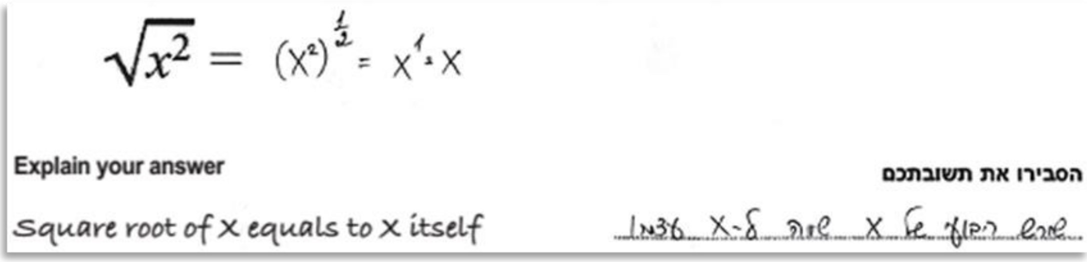


Figure 1a. Excerpt from Anna's assignment and its translation

Let us attend now to Anna's narratives that involve words for exploring whether they reflect her reliance on a multi-step realization procedure described above (see Figures 1a, 1b, and 1c for examples). When contrasted with the narratives of her classmates, two features become notable in Anna's responses: First, her usage of hybrid signifiers that combine words with symbols (i.e. "square root of x", "the expression $\sqrt{a}$" and "Expression $\sqrt{}$"). Second, while some of the students wrote that "roots do" or "are" something (see Figure 2a as an example), Anna uses the verb "equal", which is characteristic to the symbolic medium. Accordingly, I claim that her narratives are verbalizations of procedures, in which some symbols are converted into other symbols. Indeed, the narrative in Figure 1a is a summary of the simplification process that she carried out. In Figure 1b, Anna converted the verbally expressed relation between *a* and *b* into symbols, and then converted it once again into a narrative where some symbolic signifiers were copy-pasted and others were translated into words. A similar instance is evident in Figure 1c, where the radical symbol is described as turning into power. In this way, in none of the 17 tasks in the assignment, Anna exhibited an objectified talk, in which square roots are treated as extra-discursive objects, when words and symbols are tangible means for capturing the mathematical intangibles. Instead, her narratives revolved around root-related symbols as they were the objects themselves.

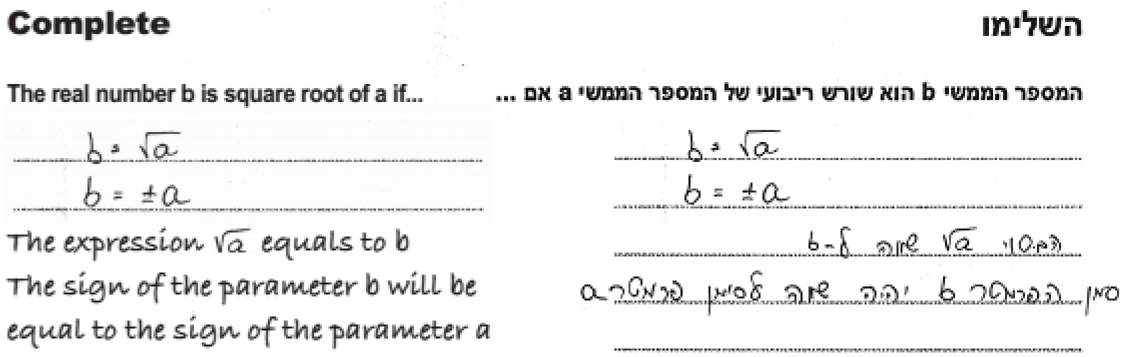

Complete
השלימו
The real number b is square root of a if...
המספר הממשי b הוא שורש ריבועי של המספר הממשי a אם ...
$b = \sqrt{a}$
$b = \pm a$
The expression $\sqrt{a}$ equals to b
The sign of the parameter b will be equal to the sign of the parameter a
הביטוי $\sqrt{a}$ שווה ל-b
סימן הפרמטר b יהיה שווה לסימן פרמטר a

Figure 2b. Excerpt from Anna's assignment and its translation

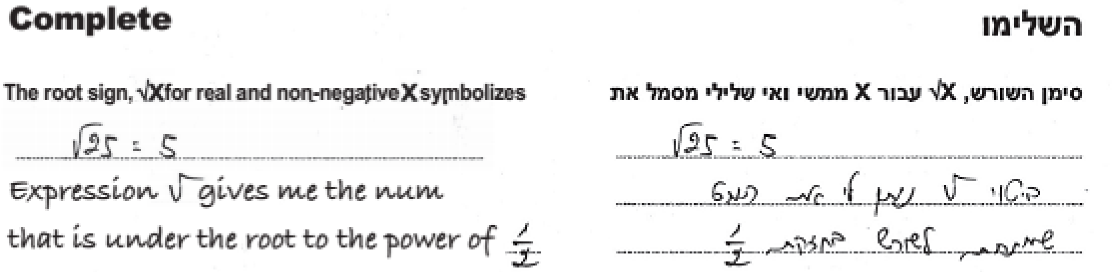

Complete
השלימו
The root sign, √X for real and non-negative X symbolizes
סימן השורש, √X עבור X ממשי ואי שלילי מסמל את
$\sqrt{25} = 5$
Expression √ gives me the num that is under the root to the power of $\frac{1}{2}$
ביטוי √ נותן לי את המספר
שמתחת לשורש בחזקת $\frac{1}{2}$

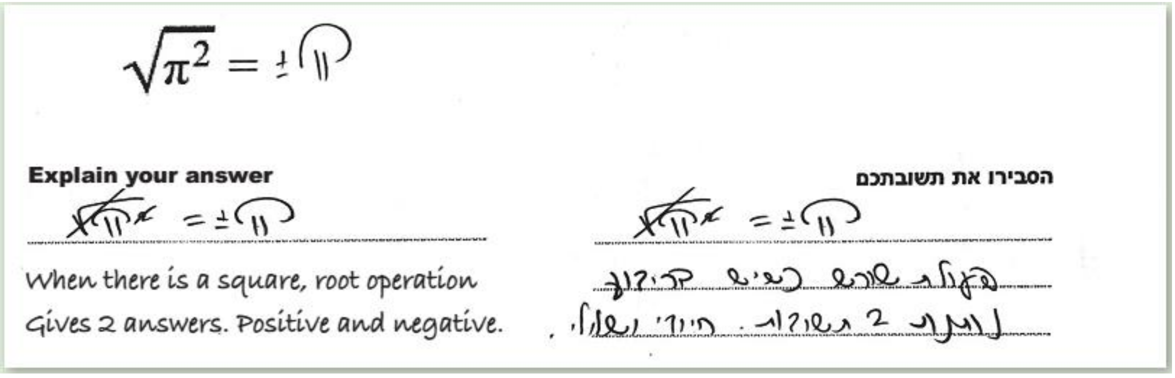
Figure 3c. Excerpt from Anna's assignment and its translation

While my interpretation of Anna's verbalizations may sound critical, her attempts to verbalize deserve appreciation. Indeed, the responses of six students to the assignment were purely symbolic. Their explanations to the simplification tasks consisted of elaborated symbolic strings that started with the assigned prompts and ended with symbols that the students provided as final answers. Furthermore, the students did not provide any response to the last two tasks, which were the most "wordy" tasks in the assignment. This preference to symbolism and avoidance of words might be interpreted as an indication of a struggle to mathematize through written texts.

Another aspect of the struggle pertains to mismatches between students' symbolic and verbal narratives. In Figure 1a, for example, Anna's narrative is concerned with *x*, while her symbolic string is prompted by x^2. In Figure 1b, the two symbolic sentences seem to contradict. Anna's verbal narrative clarifies that she realizes "$b=\pm a$" into "equal signs" rather than "equal values".

Task-dependency of Objectification

I switch now to three excerpts from Betty's assignment. Figure 2a and 2b illustrate that in the first simplification tasks she provided immediate answers. This suggests that Betty's realizations for the assigned prompts were automated. Her explanatory narratives, in turn, slightly differed in their degree of objectification. In Figure 2a, we witness her copying the assigned prompt, crossing off the power and the radical symbol, and providing a verbal narrative using “the root operation”. In the narrative, ‘root’ has the status of an adjective, which might suggest a not fully reified usage of the word. Alternatively, it may be suggested that Betty’s narrative compresses the process that she went through when engaging in the *task of explaining* her answer rather than in the *task of providing* it in the first place. Indeed, in Figure 2b, no signs of a process are evident in Betty’s symbolic writing, which aligns with her using the signifiers “the root” and “roots” as nouns. Similar traits were found in five additional responses that Betty provided.


$\sqrt{\pi^2} = \pm\pi$

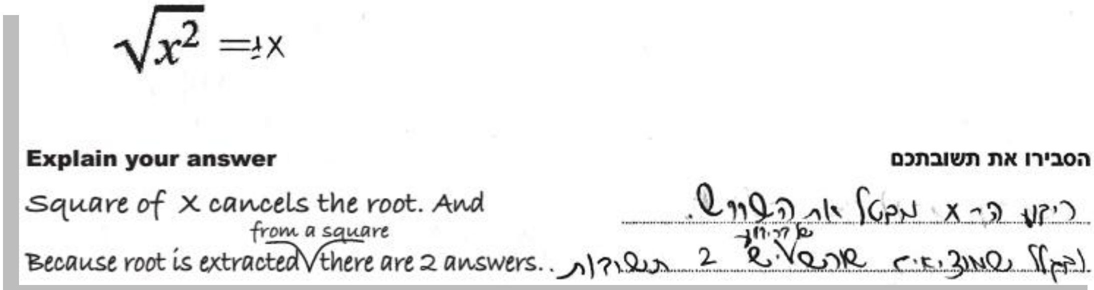
Explain your answer

הסבירו את תשובתכם

When there is a square, root operation

Gives 2 answers. Positive and negative.

Figure 2a. Excerpt from Betty’s assignment and its translation


$\sqrt{x^2} = \pm x$

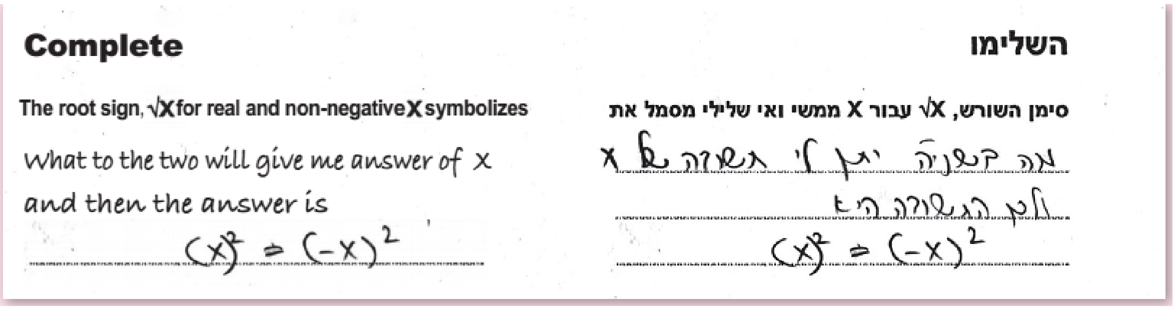
Explain your answer

הסבירו את תשובתכם

Square of X cancels the root. And

Because root is extracted from a square there are 2 answers.

Figure 2b. Excerpt from Betty’s assignment and its translation


Complete

השלימו

The root sign, $\sqrt{x}$ for real and non-negative x symbolizes

סימן השורש, $\sqrt{x}$ עבור x ממשי ואי שלילי מסמל את

What to the two will give me answer of X

and then the answer is

$(x)^2 = (-x)^2$

Figure 2c. Excerpt from Betty’s assignment and its translation

A decrease in the objectification degree is evident in Betty’s response to the task on the radical symbol (see Figure 2c). In regard to alienation, she resorts to a personal sentence for the first time in the assignment. Notably, Anna went through a similar collapse in alienation in the same task (see Figure 1c again). A collapse in reification is manifested in two ways: First, Betty uses the pronoun “what”, which releases her from realizing “$\sqrt{x}$” into a more specific signifier (compare to Anna, whose realization was “the number” with special properties). Second, Betty’s reference to a future action (i.e. “will give”) indicates a processual formulation. Yet, while Anna’s narrative captures the first step in the process –

a transition from the root symbol to the power of half – Betty's narrative is focused on the final result of the action.

Despite the mismatch between the verbal and symbolic components of her narrative (i.e. "answer of *x*" versus "$(x)^2=(-x)^2$"), Betty succeeded to complete the assigned sentence and referred to squaring – one of the defining attributes of square roots. This cannot be said about six other students, who attempted the task but were mostly concerned with the signs of x and with providing numerical examples. These responses can be interpreted as a mismatch between the task the students were assigned and the one they completed.

Summary and Discussion

In the past decade, we notice a substantial growth in the body of research that uses the commognitive framework as a lens for scrutinizing mathematics learning and teaching (e.g., Güçler, 2014; Nachlieli & Tabach, 2015; Shinno, 2018). Yet, this research rarely pays attention to the notion of objectification that has been described as key to concept formation in mathematics (Sfard, 2008). The study at hand addresses this gap by exploring how objectification can manifest itself when mature students engage in a discourse on square roots through writing.

One finding of this study pertains to task-dependency of objectification. Through a fine-grained analysis of one student's assignment (Betty), I showed that her narratives in different tasks differed in degrees of objectification. This finding resonates with the literature on mathematics lecturers and teachers. For instance, Güçler (2014) shows excerpts from a calculus classroom where a lecturer wrote objectified sentences on the blackboard but when communicated orally, they were often about processes instead of objects. This discursive move can be explained with different tasks that the lecturer was engaged in: recording "a mathematical truth" versus explaining it to the class. On the same point, Nachlieli and Tabach (2015) propose that teachers intentionally conflate their talk on objects and people in order to reduce the distance between mathematics and students. When considered together, the studies of Güçler (2014), Nachlieli and Tabach (2015), and the one reported in this paper might suggest exploring one's objectification range that she demonstrates in different tasks rather than ascribing her to a single objectification degree.

On the one hand, the suggestion might seem contradictory to Sfard (2008), who wrote that "once the project of objectification is completed, its results seem irreversible. This is why the adults seem incapable of seeing as different the things that the children cannot see as the same" (p. 141). On the other hand, Sfard's observation is a generalization of her findings from research on children's numerical discourse – probably the first mathematical discourse that children encounter in their lives. Thus, future research may be interested in exploring whether "the project of objectification" occurs differently in discourses that revolve around different mathematical objects. Indeed, it seems obvious that the discourses differ in their affordances to verbalize and symbolize, which should be reflected in how discursive objects are individualized by the learners. The language also shapes the discursive affordances, and then, this research venue might turn to be especially fruitful if teams of researchers will explore how the "same" mathematical objects are learned and taught in different countries.

Another finding of this study is concerned with collapses in objectification that were particularly evident in the tasks asking students to communicate the radical symbol and the definition of a square root. Mathematics Education discipline has accumulated a considerable body of evidence pointing to students' struggles to operate with definitions and notation (e.g., Güçler, 2014) when the lion's share of studies can be ascribed to the cognitive paradigm (e.g., Tall & Vinner, 1981). The commognitive standpoint, in turn, allows proposing that the source of the struggle is not exclusively in students "not knowing how definitions and notation work" but in a broader difficulty to discuss symbolically-signifiable objects. The participants in this study struggled to produce narratives that combine symbols and words into coherent sentences; in multiple cases, the students either wrote about symbols as they were the objects (see the assignment of Anna) or avoided the usage of words. The preference to symbolism may be interpreted as a manifestation of compressed talk where compound verbal sentences turn into laconic symbolic strings; the same compression that Sfard positions as "irreversible". Yet, there are tasks for which symbolism becomes insufficient and verbalism is required.

Another important aspect that deserves systematic scrutiny pertains to instances where "reteaching" is required because students individualized incorrect rules of a mathematical discourse. In this study, for instance, the students were divided between whether $\sqrt{x^2}$ equals x or $\pm x$, but no one suggested $|x|$ as a correct answer. So, is an objectified talk a curse or a blessing in cases where the whole discourse needs to be revised? In the same way, if the path to objectification goes through reification and alienation, how does a dis-objectification occur? While these questions might seem purely theoretical, they have high practical value. In large heterogeneous classes, a teacher can never be fully sure whether the mathematics that she teaches is new to the students or whether it distorts something well-familiar to them. Accordingly, thorough investigations of these questions may lead to research-based pedagogies that teachers can use in their daily practice.

One practical suggestion that can be made based on this study is that students should be provided with sufficient opportunities to develop different aspects of their mathematical discourse. These opportunities are inseparable from both the tasks that a teacher brings to the classroom and the classroom rules of communication that often remain tacit. For assessing these tasks and rules, the teacher may self-reflect through questions such as: "Is it sufficient for my students to specialize in symbolic manipulations for succeeding in a particular topic?"; "Are the symbolism and verbalism complementary sides of the same coin in my classroom or does one just recap the other?"; "Who usually verbalizes both orally and in writing in my classroom?". I believe that answering these questions may equip teachers with practical insights on how the mathematical discourse of their students can be improved.

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